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REVIEW

Navier-Stokes Equations. By ROGER TEMAM. North-Holland, 1977. 500 pp. $45.00.
In contrast to its bulk, the goals of this book may seem comparatively modest to scientists unfamiliar with the difficulties accompanying the careful analysis of complex partial differential equations. Readers will search in vain for provocative insights into the physics of fluids. And, excepting the last two pages, interesting flow patterns, obtained via the numerical methods expounded, are not in evidence. Nonetheless, Temam's text represents a substantial addition to the material concerning the Navier–Stokes equations available outside the journal literature.

The mathematical style of this volume is distinctly modern, reflecting some of the powerful methods in nonlinear analysis that have come to the fore over the last two decades or so. Though a complete description of certain function-analytic results particularly useful for the purposes at hand is provided, the text cannot be considered a self-contained introduction to the theory being exposited. A reader dedicated to following the arguments in detail surely needs previous experience with Sobolev spaces, functional analysis and modern numerical analysis.

Parts of Teman's tome are very reminiscent of the influential book by Lions.† Lions' stated purpose was to introduce certain methods for the resolution of nonlinear problems, using a case-study pedagogy (one of the recurring examples being the Navier–Stokes equations). In Lions text, a problem is considered 'resolved' as soon as a suitable existence theory is established. Temam, whilst overlapping considerably with Lions in the prior demands made on the reader and in his general stance toward the problems confronted has not been content with just an existence theory. Rather, he has persisted until constructive approximation schemes, complete with error bounds, are in hand. It is Temam's contention that, generally, the methodology he develops has much wider scope than just the Navier–Stokes equations. Thus one might view Temam's book as providing, in readily available form, an indication of how to proceed from the existence theories toward the more practical problem of numerical approximation of solutions of equations.

The book contains three chapters and an appendix. The first chapter, which comprises thirty per cent of the book, is devoted to the steady-state Stokes equations, posed in reasonable bounded domains in Euclidean n-space ΓR^n. That is, steady infinitesimal motions are considered, relative to a bounded container, with the no-slip boundary conditions imposed. Such an extended study of this linear system may seem an inauspicious beginning. In fact, many of the tools and techniques, introduced in the analysis of this relatively simple problem, figure decisively in the later chapters. After some preliminaries concerning function spaces, the variational formulation of the Stokes problem, due originally to LeRay, is introduced and existence and uniqueness of weak solutions is proved via a Galerkin approximation. Some further regularity results concerning these solutions are also considered. The third section of chapter one is central to the book. There, abstract approximation schema are intro-

† J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod (1969), Paris.

duced, and realized in several concrete examples, which are finite difference and finite element methods. These are then applied to the approximation of the solutions of the Stokes problem, whose existence within particular function classes has just been assured. The chapter closes with some computational algorithms, related to certain of the approximation schemes, and a glimpse of another approximation scheme, which incorporates a slight compressibility.

Chapter two confronts the full steady-state Navier–Stokes equations. The general approach is just as in chapter one. However, a technical point arises, after the Galerkin approximations are obtained. In attempting to pass to the limit, as the number of basis elements in the Galerkin approximation approaches infinity and thus assure existence of a solution to the problem, the nonlinear term is an obstacle to the straightforward application of the methods given in chapter one. To effect the desired limiting procedure, the Sobolev imbedding theorems are invoked. These theorems, which lie at the heart of much of the modern theory of nonlinear partial differential equations, give conditions for the continuous and compact inclusion of one function class within another. Using this additional tool, an existence theory, comparable to the one developed in chapter one, is established. The scene then shifts to the approximation of these solutions. For this purpose, discrete versions of the Sobolev theorems are proved and used to confirm the convergence of the approximations in the context of the schemes already introduced.

In the last section of chapter 2, the uniqueness of the steady solutions of the Navier–Stokes equations is examined. Generally, it is shown that uniqueness holds for sufficiently large values of the kinematic viscosity (or equivalently, sufficiently small body and boundary forces). In the particular context of the Taylor problem, uniqueness is shown to fail when the system experiences vigorous external forcing. The flow considered is contained between two infinitely-long cylinders whose axes coincide. The outer cylinder is held fixed, while the inner cylinder is rotated at a constant angular velocity α. It is shown that there is a critical value of α at which the flow loses uniqueness. Particularly, it is shown that there exist, in addition to a trivial radial solution, solutions which are non-trivially periodic along the axis of the cylinders. (This corresponds, in a general way, to what is observed experimentally, though the text makes no allusion to this. Nor is the crucially important stability analysis associated with this problem even mentioned.) The non-uniqueness result is confirmed using topological degree theory, the bare bones of which are reviewed for the reader's convenience.

The main act is unveiled in chapter three, where time-dependence is included. Initial conditions must be specified as well as boundary conditions on the walls of the container. To warm up the audience, Temam first considers the initial-value problem for the linearized evolution equation with no-slip boundary conditions. This proves to be a rather easy problem and a good setting in which to introduce some additional notation, spaces and methods. Just as in passing from chapter one to chapter two, serious complication sets in which the nonlinear term is included. A compactness theorem, which involves fractional order temporal derivatives, proves to be the key to further progress, though later in the chapter, the existence of weak solutions to the Navier–Stokes equations is also confirmed by a simple time-discretization. The latter method seems to yield the relevant result rather more transparently.

For two-dimensional flows, a satisfactory theory of uniqueness and further regularity is accomplished by an energy argument. For three-dimensional flows, the question of uniqueness is open in general and Temam contents himself with a sample of the type of results which can be established by the methods used effectively in two dimensions. There is also an illuminating discussion of the connexion between uniqueness of three-dimensional flows and existence of smoother solutions to the initial-value problem for the Navier–Stokes equations posed in three dimensions.

The stage is now set for the approximation of solutions of the initial-value problem by the methods introduced earlier. Of course, the temporal variable must enter as well, and so a new level of complexity arises. Here the book assumes qualities of a monograph, with a dazzling display of technique, some of it not published before. Temam is able to demonstrate the stability of various fully discrete numerical schemes and then obtained estimates for the difference between the discrete solutions and the solution of the Navier–Stokes equations being approximated. (Reflecting the state of the art in the fully-continuous problem, the results for the numerical schemes are better in two than in three dimensions.) The chapter finishes with a description of the method of fractional steps and an artificial compressibility method.

The volume itself closes with an appendix (written by Thomasset) in which an explicit implementation of one of the methods is given for some two-dimensional steady flows. The problems treated are flow in a rectangular cavity, driven by a constant flow across the open end, and steady flow between two non-concentric rotating cylinders. The method used involves non-conforming linear finite elements. Pictures of the numerically computed streamlines for these two problems appear as the book's finale.

This text developed out of a set of lecture notes, extracted from a course given by Professor Temam in 1972–1973. In many places the text follows these original notes rather closely. This is advantageous, in that the freshness and vivacity of the notes is preserved. However, it may also account for an exceptionally large number of smaller errors. For example, as far as I counted there is no less than one error every two pages, in the first hundred pages! Many of these are easily interpreted correctly. But some would be very confusing to a less than mathematically expert reader. Further evidence that the notes were not seriously revised is the inclusion of a proof, due to Lions, that the completion, in $H^1(D)$, of the divergence-free smooth vector fields with compact support in D, is equal to the elements of $H^1_0(D)$ with zero divergence, even though the proof was already called into question in Heywood's 1976 article.†

Perhaps the strongest criticism one might raise is that there is too little attempt to relate the book to fluid mechanics. Fluid mechanics does not obtain its vitality solely from existence theorems. Nor are approximation schemes of intrinsic interest to the mechanician, unless their power to cast light on interesting or controversial situations is clearly demonstrated. Certainly this was not undertaken in the text. Nor is the physical or interpretive side of the subject related to the theoretical developments. As as consequence, whilst Temam's book is a work of great mathematical virtuosity, which will be of continuing interest and importance in the realm of analysis of nonlinear partial differential equations, its impact on fluid mechanics will probably remain comparatively limited.

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† J.-G. Heywood, 'On uniqueness questions in the theory of viscous flow', *Acta Math.* 1976, **136**, 61–102.